

Statement on Competencies in Mathematics Expected of Entering College Students

Introduction

The goal of this Statement on Competencies in Mathematics Expected of Entering College Students is to provide a clear and coherent message about the mathematics that students need to know and to be able to do to be successful in college. While this is written especially for the secondary mathematics teachers, it should be useful for anyone who is concerned about the preparation of California's students for college. This represents an effort to be realistic about the skills, approaches, experiences, and subject matter that make up an appropriate mathematical background for entering college students.

The first section describes some characteristics that identify the student who is properly prepared for college courses that are quantitative in their approach. The second section describes the background in technology, such as calculators, that college students should have. The third section describes the subject matter that is an essential part of the background for all entering college students, as well as describing what is the essential background for students intending quantitative majors. Among the descriptions of subject matter there are sample problems. These are intended to clarify the descriptions of subject matter and to be representative of the appropriate level of understanding. The sample problems do not cover all of the mathematical topics identified.

No section of this Statement should be ignored. Students need the approaches, attitudes, and perspectives on mathematics described in the first section. Students need the experiences with technology described in the second section. And students need extensive skills and knowledge in the subject matter areas described in the third section. Inadequate attention to any of these components is apt to disadvantage the student in ways that impose a serious impediment to success in college. Nothing less than a balance among these components is acceptable for California's students.

The discussion in this document of the mathematical competencies expected of entering college students is predicated on the following basic recommendation:

For proper preparation for baccalaureate level course work, all students should be enrolled in a mathematics course in every semester of high school. It is particularly important that students take mathematics courses in their senior year of high school, even if they have completed three years of college preparatory mathematics by the end of their junior year. Experience has shown that students who take a hiatus from the study of mathematics in high school are very often unprepared for courses of a quantitative nature in college and are unable to continue in these courses without remediation in mathematics.

Section I. Approaches to Mathematics

This section enumerates characteristics of entering freshmen college students who have the mathematical maturity to be successful in a first college mathematics course, and in other college courses that are quantitative in their approach. These characteristics are described primarily in terms of how students approach mathematical problems. The second part of this section provides suggestions to secondary teachers of ways to present mathematics that will help their students to develop these characteristics.

Part 1. Dispositions of well-prepared students toward mathematics

Crucial to their success in college is the way in which students encounter the challenges of new problems and new ideas. From their high school mathematics courses students should have gained certain approaches, attitudes, and perspectives:

- A view that mathematics makes sense—students should perceive mathematics as a way of understanding, not as a sequence of algorithms to be memorized and applied.
- An ease in using their mathematical knowledge to solve unfamiliar problems in both concrete and abstract situations—students should be able to find patterns, make conjectures, and test those conjectures; they should recognize that abstraction and generalization are important sources of the power of mathematics; they should understand that mathematical structures are useful as representations of phenomena in the physical world; they should consistently verify that their solutions to problems are reasonable.
- A willingness to work on mathematical problems requiring time and thought, problems that aren't solved by merely mimicking examples that have already been seen—students should have enough genuine success in

- solving such problems to be confident, and thus to be tenacious, in their approach to new ones.
- A readiness to discuss the mathematical ideas involved in a problem with other students and to write clearly and coherently about mathematical topics-students should be able to communicate their understanding of mathematics with peers and teachers using both formal and natural languages correctly and effectively.
 - An acceptance of responsibility for their own learning-students should realize that their minds are their most important mathematical resource, and that teachers and other students can help them to learn but can't learn for them.
 - The understanding that assertions require justification based on persuasive arguments, and an ability to supply appropriate justifications-students should habitually ask "Why?" and should have a familiarity with reasoning at a variety of levels of formality, ranging from concrete examples through informal arguments using words and pictures to precise structured presentations of convincing arguments.
 - An openness to the use of appropriate technology, such as graphing calculators and computers, in solving mathematical problems and the attendant awareness of the limitations of this technology-students should be able to make effective use of the technology, which includes the ability to determine when technology will be useful and when it will not be useful.
 - A perception of mathematics as a unified field of study-students should see interconnections among various areas of mathematics, which are often perceived as distinct.

Part 2: Aspects of Mathematics Instruction to Foster Student Understanding and Success

There is no best approach to teaching, not even an approach that is effective for all students, or for all instructors. One criterion that should be used in evaluating approaches to teaching mathematics is the extent to which they lead to the development in the student of the dispositions, concepts, and skills that are crucial to success in college. It should be remembered that in the mathematics classroom, time spent focused on mathematics is crucial. The activities and behaviors that can accompany the learning of mathematics must not become goals in themselves-understanding of mathematics is always the goal.

While much has been written recently about approaches to teaching mathematics, as it relates to the preparation of students for success in college, there are a few aspects of mathematics instruction that merit emphasis here.

Modeling Mathematical Thinking

Students are more likely to become intellectually venturesome if it is not only expected of them, but if their classroom is one in which they see others, especially their teacher, thinking in their presence. It is valuable for students to learn with a teacher and others who get excited about mathematics, who work as a team, who experiment and form conjectures. They should learn by example that it is appropriate behavior for people engaged in mathematical exploration to follow uncertain leads, not always to be sure of the path to a solution, and to take risks. Students should understand that learning mathematics is fundamentally about inquiry and personal involvement.

Solving Problems

Problem solving is the essence of mathematics. Problem solving is not a collection of specific techniques to be learned; it cannot be reduced to a set of procedures. Problem solving is taught by giving students appropriate experience in solving unfamiliar problems, by then engaging them in a discussion of their various attempts at solutions, and by reflecting on these processes. Students entering college should have had successful experiences solving a wide variety of mathematical problems. The goal is the development of open, inquiring, and demanding minds. Experience in solving problems gives students the confidence and skills to approach new situations creatively, by modifying, adapting, and combining their mathematical tools; it gives students the determination to refuse to accept an answer until they can explain it.

Developing Analytic Ability and Logic

A student who can analyze and reason well is a more independent and resilient student. The instructional emphasis at all levels should be on a thorough understanding of the subject matter and the development of logical reasoning. Students should be asked "Why?" frequently enough that they anticipate the question, and ask it of themselves. They should be expected to construct compelling arguments to explain why, and to understand a proof comprising a significant sequence of implications. They should be expected to question and to explore why one statement follows from another. Their understandings should be challenged with questions that cause them to further examine their

reasoning. Their experience with mathematical proof should not be limited to the format of a two-column proof; rather, they should see, understand, and construct proofs in various formats throughout their course work. A classroom full of discourse and interaction that focuses on reasoning is a classroom in which analytic ability and logic are being developed.

Experiencing Mathematics in Depth

Students who have seen a lot but can do little are likely to find difficulty in college. While there is much that is valuable to know in the breadth of mathematics, a shallow but broad mathematical experience does not develop the sort of mathematical sophistication that is most valuable to students in college. Emphasis on coverage of too many topics can trivialize the mathematics that awaits the students, turn the study of mathematics into the memorization of discrete facts and skills, and divest students of their curiosity. By delving deeply into well-chosen areas of mathematics, students develop not just the self-confidence but the ability to understand other mathematics more readily, and independently.

Appreciating the Beauty and Fascination of Mathematics

Students who spend years studying mathematics yet never develop an appreciation of its beauty are cheated of an opportunity to become fascinated by ideas that have engaged individuals and cultures for thousands of years. While applications of mathematics are valuable for motivating students, and as paradigms for their mathematics, an appreciation for the inherent beauty of mathematics should also be nurtured, as mathematics is valuable for more than its utility. Opportunities to enjoy mathematics can make the student more eager to search for patterns, for connections, for answers. This can lead to a deeper mathematical understanding, which also enables the student to use mathematics in a greater variety of applications. An appreciation for the aesthetics of mathematics should permeate the curriculum and should motivate the selection of some topics.

Building Confidence

For each student, successful mathematical experiences are self-perpetuating. It is critical that student confidence be built upon genuine successes—false praise usually has the opposite effect. Genuine success can be built in mathematical inquiry and exploration. Students should find support and reward for being inquisitive, for experimenting, for taking risks, and for being persistent in finding solutions they fully understand. An environment in which this happens

is more likely to be an environment in which students generate confidence in their mathematical ability.

Communicating

While solutions to problems are important, so are the processes that lead to the solutions and the reasoning behind the solutions. Students should be able to communicate all of this, but this ability is not quickly developed. Students need extensive experiences in oral and written communication regarding mathematics, and they need constructive, detailed feedback in order to develop these skills. Mathematics is, among other things, a language, and students should be comfortable using the language of mathematics. The goal is not for students to memorize an extensive mathematical vocabulary, but rather for students to develop an ease in carefully and precisely discussing the mathematics they are learning. Memorizing terms that students don't use does not contribute to their mathematical understanding. However, using appropriate terminology so as to be precise in communicating mathematical meaning is part and parcel of mathematical reasoning.

Becoming Fluent in Mathematics

To be mathematically capable, students must have a facility with the basic techniques of mathematics. There are necessary skills and knowledge that students must routinely exercise without hesitation (see Appendix A). Mathematics is the language of the sciences, and thus fluency in this language is a basic skill. College mathematics classes require that students bring with them an ease with the standard skills of mathematics that allows them to focus on the ideas and not become lost in the details. However, this level of internalization of mathematical skills should not be mistaken for the only objective of secondary mathematics education. Student understanding of mathematics is the goal. In developing a skill, students first must develop an understanding. Then as they use the skill in different contexts, they gradually wean themselves from thinking about it deeply each time, until its application becomes routine. But their understanding of the mathematics is the map they use whenever they become disoriented in this process. The process of applying skills in varying and increasingly complex applications is one of the ways that students not only sharpen their skills, but also reinforce and strengthen their understanding. Thus, in the best of mathematical environments, there is no dichotomy between gaining skills and gaining understanding. A curriculum that is based on depth and problem solving can be quite effective in this regard provided that it focuses on appropriate areas of mathematics.

Section II. Technology

The pace at which advances are made in technology, and the surprising ways in which mathematics pedagogy and curriculum change in response to those advances, make it impossible to anticipate what technological experiences and skills students will need for success in college in the coming years. Also, the diversity of responses to technology among the college mathematics courses in California further impede the development of a clear statement on the appropriate technological background for entering college students. But the general directions are discernible. The past has shown us that scientific calculators make many problems accessible to students that previously were not because of excessive computation. More recently, we've seen that students can use the graphing capabilities of calculators to deepen their understanding of functions. And now the advent of hand-held calculators that perform symbolic algebra computations will certainly have a major impact on the instruction in algebra and more advanced courses.

From all of this, it is clear that entering college students must have availed themselves of opportunities presented by technology. The kind of graphing calculator or computer software preferred at different institutions, by different instructors, in different courses, at different times will of course vary. So, student experiences should not focus on the intricacies of a specific device so much as on the use of technology as a valuable tool in many aspects of their mathematics courses. Entering college students should have considerable experience in the following areas:

- Deciding when to use technology. Students should be able to determine what algebraic or geometric manipulations are necessary to make best use of the calculator. At the same time, they should also be able to determine for themselves when using a calculator, for example, might be advantageous in solving a problem.
- Dealing with data. Students should work on problems posed around real data and involving significant calculations. With repeated applications requiring computation, they can gain skill in estimation, approximation, and the ability to tell if a proposed solution is reasonable. Students should find opportunities to work with data in algebra, geometry, and statistics.
- Checking their calculations. Whenever possible students should use a calculator with a multi-line screen so that they are able to review the input to the calculator and to determine whether any errors have been made.

- Representing problems geometrically. Students should be able to use graphing calculators as a tool to represent functions and to develop a deeper understanding of domain, range, arithmetic operations on functions, inverse functions, and function composition.
- Experimenting, making conjectures, and finding counterexamples. Students should be comfortable using technology to check their guesses, to formulate revised guesses, and to make conjectures based on these results. They should also challenge conjectures, and find counterexamples. Where possible, they should use tools such as geometric graphing utilities to make and test geometric conjectures and to provide counterexamples.

Section III. Subject Matter

Decisions about the subject matter for secondary mathematics courses are often difficult, and are too-easily based on tradition and partial information about the expectations of the colleges. What follows is a description of mathematical areas of focus that are 1. essential for all entering college students; 2. desirable for all entering college students; 3. essential for college students to be adequately prepared for quantitative majors; and 4. desirable for college students who intend quantitative majors. This description of content will in many cases necessitate adjustments in a high school mathematics curriculum, generally in the direction of deeper study in the more important areas, at the expense of some breadth of coverage.

Sample problems have been included to indicate the appropriate level of understanding for some areas. The problems included do not cover all of the mathematical topics described, and many involve topics from several areas. Entering college students working independently should be able to solve problems like these in a short time-less than half an hour for each problem included. Students must also be able to solve more complex problems requiring significantly more time.

Part 1. Essential areas of focus for all entering college students

What follows is a summary of the mathematical subjects that are an **essential** part of the knowledge base and skill base for all students who enter higher education. Students are best served by deep mathematical experiences in these areas. This is intended as a brief compilation of the truly essential topics, as opposed to topics to which students should have been introduced but need not have mastered. The skills and content knowledge that are prerequisite to high school mathematics courses are of course still necessary for success in

college, although they are not explicitly mentioned here. Relative to traditional practice, topics and perspectives are described here as appropriate for increased emphasis (which does not mean paramount importance) and for decreased emphasis (which does not mean elimination).

* Variables, Equations, and Algebraic Expressions: Algebraic symbols and expressions; evaluation of expressions and formulas; translation from words to symbols; solutions of linear equations and inequalities; absolute value; powers and roots; solutions of quadratic equations; solving two linear equations in two unknowns including the graphical interpretation of a simultaneous solution. Increased emphasis should be placed on algebra both as a language for describing mathematical relationships and as a means for solving problems, while decreased emphasis should be placed on interpreting algebra as merely a set of rules for manipulating symbols.

The braking distance of a car (how far it travels after the brakes are applied until it comes to a stop) is proportional to the square of its speed. Write a formula expressing this relationship and explain the meaning of each term in the formula. If a car traveling 50 miles per hour has a braking distance of 105 feet, then what would its braking distance be if it were traveling 60 miles per hour?

Solve for x and give a reason for each step: $\frac{2}{3x+1} + 2 = \frac{2}{3}$

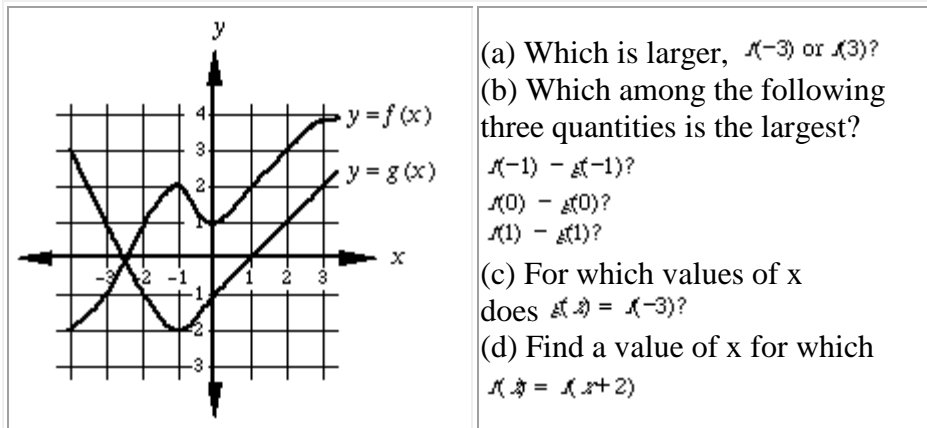
United States citizens living in Switzerland must pay taxes on their income to both the United States and to Switzerland. The United States tax is 28% of their taxable income after deducting the tax paid to Switzerland. The tax paid to Switzerland is 42% of their taxable income after deducting the tax paid to the United States. If a United States citizen living in Switzerland has a taxable income of \$75,000, how much tax must that citizen pay to each of the two countries? Find these values in as many different ways as you can; try to find ways both using and not using graphing calculators. Explain the methods you use.

* Families of Functions and Their Graphs: Applications; linear functions; quadratic and power functions; exponential functions; roots; operations on functions and the corresponding effects on their graphs; interpretation of graphs; function notation; functions in context, as models for data. Increased emphasis should be placed on various representations of functions—using graphs, tables, variables, words—and on the interplay among the graphical and other representations, while decreased emphasis should be placed on repeated manipulations of algebraic expressions.

Car dealers use the "rule of thumb" that a car loses about 30% of its value each year.

Suppose that you bought a new car in December 1995 for \$20,000. According to this "rule of thumb," what would the car be worth in December 1996? In December 1997? In December 2005?

Develop a general formula for the value of the car t years after purchase.



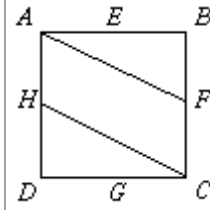
Find a quadratic function of x that has zeroes at $x = -1$ and $x = 2$.

Find a cubic function of x that has zeroes at $x = -1$ and $x = 2$ and nowhere else.

* Geometric Concepts: Distances, areas, and volumes, and their relationship with dimension; angle measurement; similarity; congruence; lines, triangles, circles, and their properties; symmetry; Pythagorean Theorem; coordinate geometry in the plane, including distance between points, midpoint, equation of a circle; introduction to coordinate geometry in three dimensions; right angle trigonometry. Increased emphasis should be placed on developing an understanding of geometric concepts sufficient to solve unfamiliar problems and an understanding of the need for compelling geometric arguments, while decreased emphasis should be placed on memorization of terminology and formulas.

A contemporary philosopher wrote that in 50 days the earth traveled approximately 40 million miles along its orbit and that the distance between the positions of the earth at the beginning and the end of the 50 days was approximately 40 million miles. Discuss any errors you can find in these conclusions or explain why they seem to be correct. You may approximate the earth's orbit by a circle with radius 93 million miles.

ABCD is a square and the midpoints of the sides are E, F, G, and H. $AB = 10$ in. Use at least two different methods to find the area of parallelogram AFCH.



Two trees are similar in shape, but one is three times as tall as the other.

If the smaller tree weighs two tons, how much would you expect the larger tree to weigh?

Suppose that the bark from these trees is broken up and placed into bags for landscaping uses. If the bark from these trees is the same thickness on the smaller tree as the larger tree, and if the larger tree yields 540 bags of bark, how many bags would you expect to get from the smaller tree?

An $8\frac{1}{2}$ in. by 11 in. sheet of paper can be rolled lengthwise to make a cylinder, or it can be rolled widthwise to make a different cylinder.

Without computing the volumes of the two cylinders, predict which will have the greater volume, and explain why you expect that.

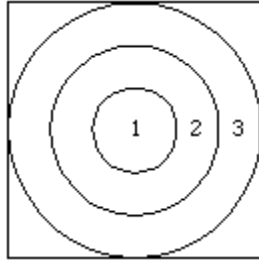
Find the volumes of the two cylinders to see if your prediction was correct.

If the cylinders are to be covered top and bottom with additional paper, which way of rolling the cylinder will give the greater total surface area?

- **Probability:** Counting (permutations and combinations, multiplication principle); sample spaces; expected value; conditional probability; area representations of probability. Increased emphasis should be placed on a conceptual understanding of discrete probability, while decreased emphasis should be placed on aspects of probability that involve memorization and rote application of formulas.

If you take one jelly bean from a large bin containing 10 lbs. of jelly beans, the chance that it is cherry flavored is 20%. How many more pounds of cherry jelly beans would have to be mixed into the bin to make the chance of getting a cherry one 25%?

A point is randomly illuminated on a computer game screen that looks like the figure shown below.



The radius of the inner circle is 3 inches; the radius of the middle circle is 6 inches; the radius of the outer circle is 9 inches.

What is the probability that the illuminated point is in region 1?

What is the probability that the illuminated point is in region 1 if you know that it isn't in region 2?

A fundraising group sells 1000 raffle tickets at \$5 each. The first prize is an \$1,800 computer. Second prize is a \$500 camera and the third prize is \$300 cash. What is the expected value of a raffle ticket?

Five friends line up at a movie theater. What is the probability that Mary and Mercedes are standing next to each other?

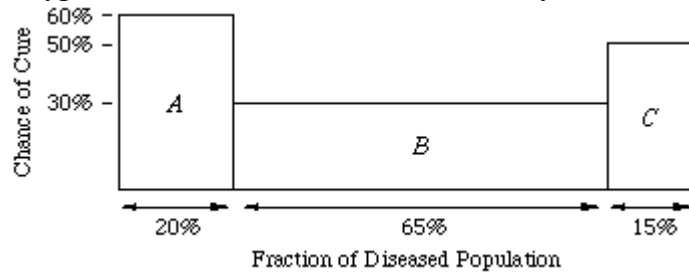
- **Data Analysis and Statistics:** Presentation and analysis of data; mean, median and standard deviation; representative samples; using lines to fit data and make predictions. Increased emphasis should be placed on organizing and describing data and making predictions based on the data, with common sense as a guide, while decreased emphasis should be placed on aspects of statistics that are learned as algorithms without an understanding of the underlying ideas.

The table at the right shows the population of the USA in each of the last five censuses. Make a scatter plot of this data and draw a line on your scatter plot that fits this data well. Find an equation for your line, and use this equation to predict what the population might be in the year 2000. Plot that predicted point on your graph and see if it seems reasonable. What is the slope of your line? Write a sentence that describes to someone who might not know about graphs and lines what the meaning of the slope is in terms involving the USA population.

Year	USA Population (millions)
1950	152.3
1960	180.7
1970	205.1
1980	227.7
1990	249.9

The results of a study of the effectiveness of a certain treatment for a

blood disease are summarized in the chart shown below. The blood disease has three types, A, B, and C. The cure rate for each of the types is shown vertically on the chart. The percentage of diseased persons with each type of the disease is shown horizontally on the same chart.



For which type of the disease is the treatment most effective?

From which type of the disease would the largest number of patients be cured by the treatment?

What is the average cure rate of this treatment for all of the persons with the disease?

Find the mean and standard deviation of the following seven numbers:

4 12 5 6 8 5 9

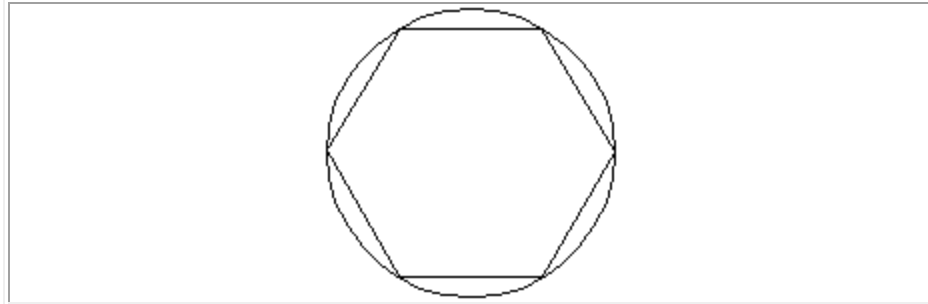
Make up another list of seven numbers with the same mean and a smaller standard deviation.

Make up another list of seven numbers with the same mean and a larger standard deviation.

- Argumentation and Proof: Mathematical implication; hypotheses and conclusions; direct and indirect reasoning; inductive and deductive reasoning. Increased emphasis should be placed on constructing and recognizing valid mathematical arguments, while decreased emphasis should be placed on mathematical proofs as formal exercises.

Select any odd number, then square it, and then subtract one. Must the result always be even? Write a convincing argument.

Use the perimeter of a regular hexagon inscribed in a circle to explain why $[\pi] > 3$.



Does the origin lie inside of, outside of, or on the geometric figure whose equation is $x^2 + y^2 - 10x + 10y - 1 = 0$? Explain your reasoning.

Part 2. Desirable areas of focus for all entering college students

What follows is a brief summary of some of the mathematical subjects that are a **desirable** part of the mathematical experiences for all students who enter higher education. No curriculum would include study in all of these areas, as that would certainly be at the expense of opportunities for deep explorations in selected areas. But these areas provide excellent contexts for the approaches to teaching suggested in Section I, and any successful high school mathematics program will include some of these topics. The emphasis here is on enrichment and on opportunities for student inquiry.

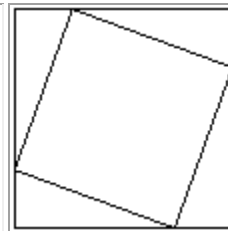
- Discrete Mathematics: Graph theory; coding theory; voting systems; game theory; decision theory.
- Sequences and Series: Geometric and arithmetic sequences and series; the Fibonacci sequence; recursion relations.
- Geometry: Transformational geometry, including rotations, reflections, translations, and dilations; tessellations; solid geometry; three-dimensional coordinate geometry, including lines and planes.
- Number Theory: Prime numbers; prime factorization; rational and irrational numbers; triangular numbers; Pascal's triangle; Pythagorean triples.

Part 3. Essential areas of focus for students in quantitative majors

What follows is a brief summary of the mathematical subjects that are an **essential** part of the knowledge base and skill base for students to be adequately prepared for quantitative majors. Students are best served by deep mathematical experiences in these areas. The skills and content knowledge listed above as essential for all students entering college are of course also essential for these students—moreover, students in quantitative majors must have a deeper understanding of and a greater facility with those areas.

- Variables, Equations, and Algebraic Expressions: Solutions to systems of equations, and their geometrical interpretation; solutions to quadratic equations, both algebraic and graphical; the correspondence between roots and factors of polynomials; the binomial theorem.

In the figure shown to the right, the area between the two squares is 11 square inches. The sum of the perimeters of the two squares is 44 inches. Find the length of a side of the larger square.



Determine the middle term in the binomial expansion of $\left(x - \frac{2}{x}\right)^{10}$.

- Functions: Logarithmic functions, their graphs, and applications; trigonometric functions of real variables, their graphs, properties including periodicity, and applications; basic trigonometric identities; operations on functions, including addition, subtraction, multiplication, reciprocals, division, composition, and iteration; inverse functions and their graphs; domain and range.

Which of the following functions are their own inverses? Use at least two different methods to answer this, and explain your methods.

$$f(x) = \frac{2}{x}$$

$$g(x) = x^3 + 4$$

$$h(x) = \frac{2 + \ln(x)}{2 - \ln(x)}$$

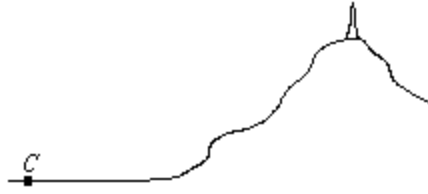
$$k(x) = \sqrt{\frac{x^2 + 1}{x^2 - 1}}$$

Scientists have observed that living matter contains, in addition to Carbon, C12, a fixed percentage of a radioactive isotope of Carbon, C14. When the living material dies, the amount of C12 present remains constant, but the amount of C14 decreases exponentially with a half life of 5,550 years. In 1965, the charcoal from cooking pits found at a site in Newfoundland used by Vikings was analyzed and the percentage of C14 remaining was found to be 88.6%. What was the approximate date of this Viking settlement?

Find all quadratic functions of x that have zeroes at $x = -1$ and $x = 2$.

Find all cubic functions of x that have zeroes at $x = -1$ and $x = 2$ and nowhere else.

A cellular phone system relay tower is located atop a hill. You have a transit and a calculator. You are standing at point C. Assume that you have a clear view of the base of the tower from point C, that C is at sea level, and that the top of the hill is 2000 ft. above sea level.



Describe a method that you could use for determining the height of the relay tower, without going to the top of the hill.

Next choose some values for the unknown measurements that you need in order to find a numerical value for the height of the tower, and find the height of the tower.

- Geometric Concepts: Two- and three-dimensional coordinate geometry; locus problems; polar coordinates; vectors; parametric representations of curves.

Find any points of intersection (first in polar coordinates and then in rectangular coordinates) of the graphs of $r = 1 + \sin \theta$ and the circle of radius $\frac{3}{2}$ centered about the origin. Verify your solutions by graphing the curves.

Find any points of intersection (first in polar coordinates and then in rectangular coordinates) of the graphs of $r = 1 + \sin \theta$ and the line with slope 1 that passes through the origin. Verify your solutions by graphing the curves.

Marcus is in his back yard, and has left his stereo and a telephone 24 feet apart. He can't move the stereo or the phone, but he knows from experience that in order to hear the telephone ring, he must be located so that the stereo is at least twice as far from him as the phone. Draw a diagram with a coordinate system chosen, and use this to find out where Marcus can be in order to hear the phone when it rings.

A box is twice as high as it is wide and three times as long as it is wide. It just fits into a sphere of radius 3 feet. What is the width of the box?

- Argumentation and Proof: Mathematical induction and formal proof. Attention should be paid to the distinction between plausible, informal reasoning and complete, rigorous demonstration.

Select any odd number, then square it, and then subtract one. Must the result always be divisible by 4? Must the result always be divisible by 8? Must the result always be divisible by 16? Write convincing arguments or give counterexamples.

The midpoints of a quadrilateral are connected to form a new quadrilateral. Prove that the new quadrilateral must be a parallelogram.

In case the first quadrilateral is a rectangle, what special kind of parallelogram must the new quadrilateral be? Explain why your answer is correct for any rectangle.

Part 4. Desirable areas of focus for students in quantitative majors

What follows is a brief summary of some of the mathematical subjects that are a **desirable** part of the mathematical experiences for students who enter higher education with the possibility of pursuing quantitative majors. No curriculum would include study in all of these areas, as that would certainly be at the expense of opportunities for deep explorations in selected areas. But these areas each provide excellent contexts for the approaches to teaching suggested in Section I. The emphasis here is on enrichment and on opportunities for student inquiry.

- Vectors and Matrices: Vectors in the plane; complex numbers and their arithmetic; vectors in space; dot and cross product, matrix operations and applications.
- Probability and Statistics: Continuous distributions; binomial distributions; fitting data with curves; regression; correlation; sampling.
- Conic Sections: Representations as plane sections of a cone; focus-directrix properties; reflective properties.
- Non-Euclidean Geometry: History of the attempts to prove Euclid's parallel postulate; equivalent forms of the parallel postulate; models in a circle or sphere; seven-point geometry.
- Calculus (*Students should take calculus only if they have demonstrated a mastery of algebra, geometry, trigonometry, and coordinate geometry. Their calculus course should be treated as a college level course and should prepare them to take one of the College Board's Advanced Placement Examinations. A joint statement from the Mathematical Association of America and the National Council of Teachers of Mathematics concerning calculus in secondary schools is included as Appendix B.*)

Comments on Implementation

Students who are ready to succeed in college will have become prepared throughout their primary and secondary education, not just in their college preparatory high school classes. Concept and skill development in the high school curriculum should be a deliberately coordinated extension of the elementary and middle school curriculum. This will require some changes, and some flexibility, in the planning and delivery of curriculum, especially in the first three years of college preparatory mathematics. For example, student understanding of probability and data analysis will be based on experiences that began when they began school, where they became accustomed to performing experiments, collecting data, and presenting the data. This is a more substantial and more intuitive understanding of probability and data analysis than one based solely on an axiomatic development of probability functions on a sample space, for example. It must be noted that inclusion of more study of data analysis in the first three years of the college preparatory curriculum, although an extension of the K-8 curriculum, will be at the expense of some other topics. The general direction, away from a broad but shallow coverage of algebra and geometry topics, should allow opportunities for this.

Appendix A

What follows is a collection of skills that students must routinely exercise without hesitation in order to be prepared for college work. These are intended as indicators—students who have difficulty with many of these skills are significantly disadvantaged and are apt to require remediation in order to succeed in college courses. This list is not exhaustive of the basic skills. This is also **not** a list of skills that are sufficient to ensure success in college mathematical endeavors.

The absence of errors in student work is not the litmus test for mathematical preparation. Many capable students will make occasional errors in performing the skills listed below, but they should be in the habit of checking their work and thus readily recognize these mistakes, and should easily access their understanding of the mathematics in order to correct them.

1. Perform arithmetic with signed numbers, including fractions and percentages.
2. Combine like terms in algebraic expressions.

3. Use the distributive law for monomials and binomials.
4. Factor monomials out of algebraic expressions.
5. Solve linear equations of one variable.
6. Solve quadratic equations of one variable.
7. Apply laws of exponents.
8. Plot points that are on the graph of a function.
9. Given the measures of two angles in a triangle, find the measure of the third.
10. Find areas of a right triangles.
11. Find and use ratios from similar triangles.
12. Given the lengths of two sides of a right triangle, find the length of the third side.

Appendix B

Calculus in the Secondary School

To: Secondary School Mathematics Teachers

From: The Mathematical Association of America

The National Council of Teachers of Mathematics

Date: September 1986

Re: Calculus in the Secondary School

Dear Colleagues:

A single variable calculus course is now well established in the 12th grade at many secondary schools, and the number of students enrolling is increasing substantially each year. In this letter we would like to discuss two problems that have emerged.

The first problem concerns the relationship between the calculus course offered in high school and the succeeding calculus courses in college. The Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM) recommend that the calculus course offered in the 12th grade should be treated as a college-level course. The expectation should be that a substantial majority of the students taking the course will master the material and will not then repeat the subject upon entrance to college. Too many students now view their 12th grade calculus course as an introduction to calculus with the expectation of repeating the material in college. This causes an undesirable attitude on the part of the student both in secondary school and in college. In secondary school all too often a student may feel "I don't have to master this material now, because I can repeat it later;" and in college, "I don't have to study this subject too seriously, because I have already seen most of the ideas." Such students typically have considerable difficulty later on as they proceed further into the subject matter.

MAA and NCTM recommend that all students taking calculus in secondary school who are performing satisfactorily in the course should expect to place out of the comparable college calculus course. Therefore, to verify appropriate placement upon entrance to college, students should either take one of the Advanced Placement (AP) Calculus Examinations of the College Board, or take a locally-administered college placement examination in calculus. Satisfactory performance on an AP examination carries with it college credit at most universities.

A second problem concerns preparation for the calculus course. MAA and NCTM recommend that students who enroll in a calculus course in secondary school should have demonstrated mastery of algebra, geometry, trigonometry, and coordinate geometry. This means that students should have at least four full years of mathematical preparation beginning with the first course in algebra. The advanced topics in algebra, trigonometry, analytic geometry, complex numbers, and elementary functions studied in depth during the fourth year of preparation are critically important for students' later courses in mathematics.

It is important to note that at present many well-prepared students take calculus in the 12th grade, place out of the comparable course in college, and do well in succeeding college courses. Currently the two most common methods for preparing students for a college-level calculus course in the 12th grade are to begin the first algebra course in the 8th grade or to require students to take second year algebra and geometry concurrently. Students beginning with algebra in the 9th grade who take only one mathematics course each year in secondary school should not expect to take calculus in the 12th grade. Instead,

they should use the 12th grade to prepare themselves fully for calculus as freshmen in college.

We offer these recommendations in an attempt to strengthen the calculus program in secondary schools. They are not meant to discourage the teaching of college-level calculus in the 12th grade to strongly prepared students.